

# Computing bi-tangents for transmission belts

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## Abstract

In this note, we determine the bi-tangents of two rotated ellipses, and we compute the coordinates of their points of tangency. For these purposes, we develop two approaches. The first one is an analytical approach in which we compute analytically the equations of the bi-tangents. This approach is valid only for some cases. The second one is geometrical and is based on the determination of the normal vector to the tangent line. This approach turns out to be more robust than the first one and is valid for any configuration of ellipses.

**Keywords:** bi-tangents, rotated ellipses, analytical approach, geometrical approach

## 1 Introduction

This note presents the results obtained at Semaine d'études maths-entreprise (SEME), organized by "l'Agence pour les Mathématiques en Interaction avec les Entreprises" (AMIES) and "Laboratoire de Mathématiques de Besançon" (LMB), and which took place in Besançon from 20 to 24 May 2019.

Supervised by Professor Franz Chouly and Professor Alexei Lozinski, our study aimed to compute the equations of the bi-tangents of two rotated ellipses and their points of tangency. This topic was proposed by Sébastien Passos, a PhD student at PSA group.

The main motivation of this study is to improve the automobile engine by designing a more adequate belt synchronous. The method is intended to be integrated into a tool for simulating the angular dynamics of synchronous belt transmissions (or automobile distribution facades) incorporating non-circular pulleys. Usually, "multi-lobed" pulleys i.e. having  $N$  lobes are used in an automobile. These pulleys are a passive control solution to reduce the angular vibrations impacting the facades. If left unchecked, these angular vibrations degrade facade performance (power loss, durability, reliability, noise) and can be caused by various excitation sources (crankshaft acyclic, fluctuating load torque on driven pulleys).

This configuration is modeled as follows: Consider an angle  $\theta$  which represents the angle from the positive horizontal axis to the pulley's major axis. For any two angles  $\theta_1$  and  $\theta_2$ , the geometric strand

connecting the pulley 1 to the pulley 2 corresponds to the segment  $[BA]$  in  $B$  (resp. in  $A$ ), the local tangent to the profile of the pulley 1 (or 2) is collinear with  $[BA]$ , as shown in Figure 1.

The first issue is the determination of the equations of the existing bi-tangents (crossed/uncrossed) and the coordinates of the intersecting points in the case of two different pulleys both with elliptical profile and whatever their respective angles of orientation are. The second one is the extension of the study to the case of any convex profiles “sufficiently” regular (e.g. non-flyers pulleys).

To be integrated with the simulation tools, the method must be robust, ideally generalist (independent of the number of lobes) and have short computation times. Emphasize that there is no explicit analytical solution to the problem. This is the whole complexity of the subject (1; 2; 3). We must develop a numerical method to answer this difficulty.

This note is outlined as follows. Section 2 presents the definitions and notations. The analytical approach is discussed in section 3, while the geometrical one is developed in section 4. The last section is devoted to present the codes for both approaches.

## 2 Definitions and notations

Our aim, in this work, is to find the bi-tangents for two rotated ellipses and their tangent points. The possible configurations for bi-tangents of two ellipses are four, more precisely, one can characterize them as follows: the outer bi-tangents from both sides right  $[A_1, B_2]$  and left  $[A_2, B_1]$  and the separating bi-tangents from both sides right  $[A_1, A_2]$  and left  $[B_1, B_2]$ , as shown in Figure 1, we refer to [Lemma 1, (3)] for more details. In the following, we assume that the ellipses are disjoint.

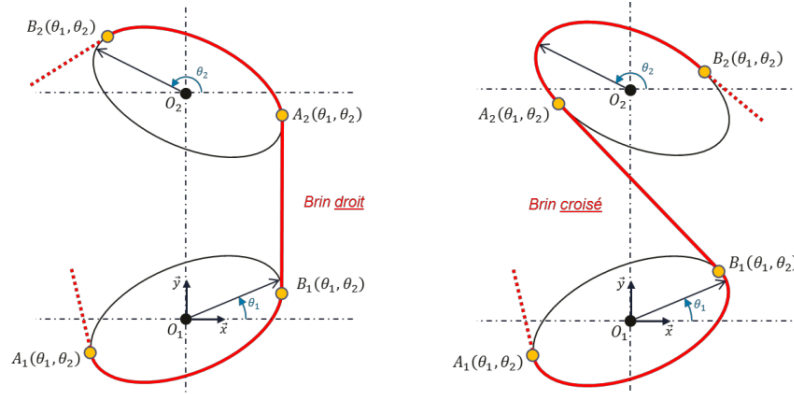


Figure 1: Possible configurations for bi-tangents.

Thus, one has to write the cartesian equation for a rotated ellipse with a given angle  $\theta$ . Analytically, the equation of an ellipse  $E$  is given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

where the shape parameters are  $a$  the semi-major axis,  $b$  the semi-minor axis and  $(h, k)$  the ellipse center. Additionally, the parametric representation of an ellipse is as follows

$$\begin{cases} x(t) = a \cos(t) + h \\ y(t) = b \sin(t) + k \end{cases} \quad (1)$$

where the parameter  $t \in [0, 2\pi[$ .

Currently, one can consider the rotation matrix  $R$  with a given angle  $\theta$ , which represents the angle from

the positive horizontal axis to the ellipse's major axis

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

By multiplying equation (1) with the rotation matrix  $R$ , one gets the parametric equation for a rotated ellipse as follows

$$\begin{cases} \frac{dx}{dt} = a \cos(\theta) \cos(t) - b \sin(\theta) \sin(t) \\ \frac{dy}{dt} = a \sin(\theta) \cos(t) + b \cos(\theta) \sin(t) \end{cases}$$

Finally, one gets the cartesian equation for a rotated ellipse with an angle  $\theta$  as follows

$$\frac{((x-h)\cos(\theta) + (y-k)\sin(\theta))^2}{a^2} + \frac{((x-h)\sin(\theta) - (y-k)\cos(\theta))^2}{b^2} = 1$$

### 3 Analytical approach

In this section, we develop an analytical approach which is based on the analytical computation of the equations of the bi-tangents. Firstly, we express two equations whose solutions correspond to a slope and a point for every one of the four tangent lines. With this at hand, we compute the coordinates of the eight points of tangency as well as the bi-tangent equations. Then, some examples are presented to illustrate the algorithm efficiency.

#### 3.1 Bi-tangents equations

The cartesian equations for two ellipses:  $E_1((a_1, b_1), \theta_1)$  centered at the origin and  $E_2((a_2, b_2), \theta_2)$  centered at the point  $(0, k)$ , as the considered case in Figure 1.

$$(E_1) : \frac{(x \cos(\theta_1) + y \sin(\theta_1))^2}{a_1^2} + \frac{(x \sin(\theta_1) - y \cos(\theta_1))^2}{b_1^2} = 1$$

$$(E_2) : \frac{(x \cos(\theta_2) + (y-k) \sin(\theta_2))^2}{a_2^2} + \frac{(x \sin(\theta_2) - (y-k) \cos(\theta_2))^2}{b_2^2} = 1$$

Consider  $T$  as a tangent line to both ellipses  $E_1$  and  $E_2$ , with equation

$$T : y = mx + p \tag{2}$$

$T$  is tangent to  $E_1$ , if and only if, it intersects  $E_1$  in a single point. By substituting the equation (2) into the equation of  $E_1$ , it yields a polynomial of degree 2 depending in  $x$  and given by

$$P_1 : \alpha_1(m, p) x^2 + \beta_1(m, p) x + \gamma_1(m, p) = 0$$

where

$$\alpha_1(m, p) = [(b_1^2 + a_1^2 m^2) \cos^2((\theta_1)) + (b_1^2 m^2 + a_1^2) \sin^2((\theta_1)) + 2m(b_1^2 - a_1^2) \cos((\theta_1)) \sin((\theta_1))]$$

$$\beta_1(m, p) = [2p(b_1^2 m \sin^2((\theta_1)) + a_1^2 m \cos^2((\theta_1)) + (b_1^2 - a_1^2) \cos((\theta_1)) \sin((\theta_1))]$$

$$\gamma_1(m, p) = p^2(b_1^2 \sin^2((\theta_1)) + a_1^2 \cos^2((\theta_1))) - a_1^2 b_1^2$$

The polynomial  $P_1$  has a single root, if and only if, its discriminant is null, that is

$$\Delta_1(m, p) = \beta_1(m, p)^2 - 4 \alpha_1(m, p) \gamma_1(m, p) \tag{3}$$

$T$  is tangent to  $E_2$ , if and only if, it intersects  $E_2$  in a single point. By substituting the equation (2) into the equation of  $E_2$ , it yields a polynomial of degree 2 depending in  $x$  and given by

$$P_2 : \alpha_2(m, p) x^2 + \beta_2(m, p) x + \gamma_2(m, p) = 0$$

where

$$\alpha_2(m, p) = [(b_2^2 + a_2^2 m^2) \cos^2(\theta_2) + (b_2^2 m^2 + a_2^2) \sin^2(\theta_2) + 2m(b_2^2 - a_2^2) \cos(\theta_2) \sin(\theta_2)]$$

$$\beta_2(m, p) = [2(p - k)(b_2^2 m \sin^2(\theta_2) + a_2^2 m \cos^2(\theta_2) + (b_2^2 - a_2^2) \cos(\theta_2) \sin(\theta_2))]$$

$$\gamma_2(m, p) = (p - k)^2 (b_2^2 \sin^2(\theta_2) + a_2^2 \cos^2(\theta_2)) - a_2^2 b_2^2$$

The polynomial  $P_2$  has a single root, if and only if, its discriminant is null, that is

$$\Delta_2(m, p) = \beta_2(m, p)^2 - 4 \alpha_2(m, p) \gamma_2(m, p) \quad (4)$$

By Eq.3 and Eq.4, one has two equations with two variables  $(m, p)$  to be determined.

$$\begin{cases} \Delta_1(m, p) = 0 \\ \Delta_2(m, p) = 0 \end{cases}$$

Analytically, it is difficult to solve this system. So, we use Maple which performs formal computations to solve it. Below, we present the algorithm and some examples to verify our calculations.

## 3.2 Numerical results

### 3.2.1 Algorithm

The algorithm based on the analytical approach introduced above needs the following inputs

- The value of  $k$  which determines the position of the second ellipse.
- The lengths of the axes of the two ellipses  $a_1, b_1$  and  $a_2, b_2$ .
- The angles of rotation of the ellipses  $\theta_1$  and  $\theta_2$ .

Since the two discriminants  $\Delta_1$  and  $\Delta_2$  are quadratic in  $m$  and  $p$ , the first thing the algorithm does is that it solves the system  $\{\Delta_1 = 0, \Delta_2 = 0\}$  and gives the four values of  $m$  as well as  $p$  corresponding to the crossed and the uncrossed bi-tangents. Using the polynomial  $P_1$  (respectively  $P_2$ ) resulting from the fact that any of the four tangents intersects the ellipse  $E_1$  (respectively  $E_2$ ) in a single point, we can express the abscissa of the intersection point of any of the four tangents with the ellipse  $E_1$  (respectively  $E_2$ ) in terms of  $m$  and  $p$ . Now, substituting the four values of  $m$  and  $p$ , the algorithm obtains the abscissas of the eight tangency points. Using the equations of the tangents, it computes the corresponding ordinates. Finally, it draws the two ellipses with the two bi-tangents. The whole algorithm is presented in detail in Appendix A.

### 3.2.2 Examples

The above algorithm succeeds to perform the required computations for many cases and for different values of the angles of rotations. For the following two examples, the values between the parentheses represents the coordinates of the eight points of tangency.

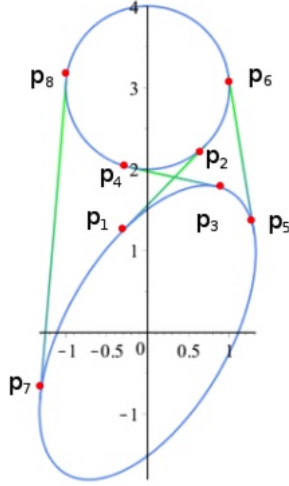


Figure 2: Example 3.1.

**Example 3.1.**

$$E_1 : a_1 = 2, b_1 = 1, \theta_1 = \frac{\pi}{3}$$

$$E_2 : a_2 = 1, b_2 = 1, \theta_2 = 0$$

$$k = 3$$

$$\begin{array}{ll} p_1 : (-0.3283773835, 1.220781461) & p_2 : (0.7192173073, 2.305214805) \\ p_3 : (0.8518931535, 1.789013789) & p_4 : (-0.2145610073, 2.023289409) \\ p_5 : (1.304873140, 1.217188112) & p_6 : (0.9868874996, 3.161409619) \\ p_7 : (-1.316373896, -0.827445267) & p_8 : (-0.9966721396, 3.081514733) \end{array}$$

**Example 3.2.**

$$E_1 : a_1 = 2, b_1 = 1, \theta_1 = \frac{\pi}{6}$$

$$E_2 : a_2 = 1, b_2 = \frac{1}{2}, \theta_2 = \frac{\pi}{6}$$

$$k = 3$$

$$\begin{array}{ll} p_1 : (-1.499999999, 0.015828568) & p_2 : (0.7499999997, 2.992085717) \\ p_3 : (1.500000000, 1.214940664) & p_4 : (-0.7500000003, 2.392529668) \\ p_5 : (-1.771690969, -0.503023819) & p_6 : (-0.8858454846, 2.748488090) \\ p_7 : (1.771690970, 0.913280228) & p_8 : (0.8858454846, 3.456640116) \end{array}$$

As mentioned before, this approach is inefficient for some cases. For instance, for the case considered below, it fails in giving any information about the bi-tangents as well as the points of tangency.

**Counter Example 3.1.**

$$E_1 : a_1 = 2, b_1 = 1, \theta_1 = \frac{\pi}{4}$$

$$E_2 : a_2 = 2, b_2 = 1, \theta_2 = \frac{\pi}{3}$$

$$k = 3$$

Moreover, in some cases the time required to perform the computations is not immediate as it requires a few seconds. For that, the next section is devoted to present another approach which covers all the possible cases and gives the required results immediately.

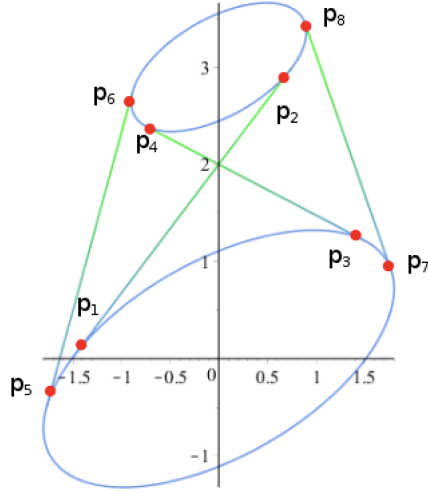


Figure 3: Example 3.2.

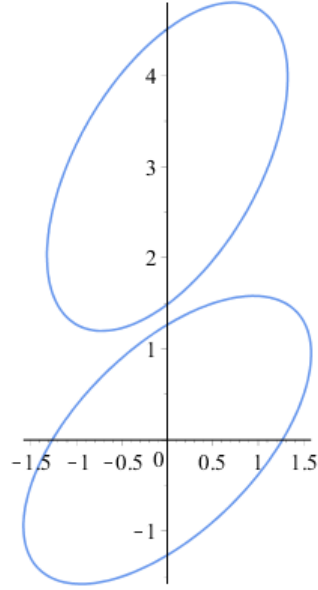


Figure 4: Counter Example 3.1.

## 4 Geometrical approach

In this section, we present a geometrical approach which turns out to be more robust than the analytical one and gives results for all possible configurations of ellipses.

## 4.1 Matrix form

The cartesian equation for an ellipse  $E_1$  is given by

$$\frac{x_1^2}{a_1^2} + \frac{y_1^2}{b_1^2} = 1 \quad (5)$$

with  $a_1 > 0$  and  $b_1 > 0$ . Consider a point  $X_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  in  $\mathbb{R}^2$  and a matrix  $M_1$  as follows

$$M_1 = \begin{pmatrix} \frac{1}{a_1^2} & 0 \\ 0 & \frac{1}{b_1^2} \end{pmatrix}$$

We write Eq.5 in a matrix form as follows

$$\phi_1(X_1) = X_1^t M_1 X_1 = 1 \quad (6)$$

Now, we are looking for the expression of  $X_1$  that verifies Eq.6. For that purpose, we consider  $n$  as a normal vector to the tangent line  $T$  of the ellipse  $E_1$  at  $X_1$  (see Figure 5).

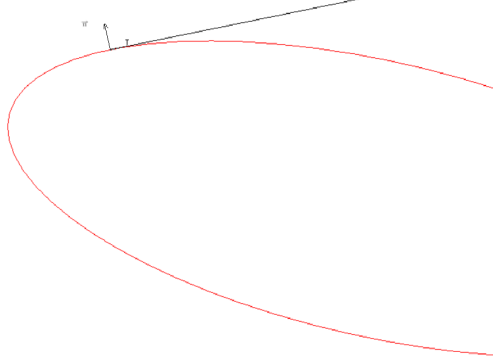


Figure 5: Normal vector  $n$  to the tangent line  $T$ .

Then, we get from the collinearity of  $n$  to the gradient of the quadratic form  $\phi_1(X_1)$  the following relation

$$\frac{1}{2} \nabla \phi_1(X_1) = M_1 X_1 = \alpha_1 n, \quad (7)$$

where the parameter  $\alpha_1$  is a positive constant. By using that  $M_1$  is an invertible matrix and from Eq.7, we obtain

$$X_1 = \alpha_1 M_1^{-1} n \quad (8)$$

By substituting Eq.8 into Eq.6 and using that  $M_1$  is a diagonal matrix, one has

$$1 = X_1^t M_1 X_1 = (\alpha_1 M_1^{-1} n)^t M_1 (\alpha_1 M_1^{-1} n) = \alpha_1^2 n^t M_1^{-t} n = \alpha_1^2 n^t M_1^{-1} n,$$

and from this we get

$$\alpha_1 = \frac{1}{\sqrt{n^t M_1^{-1} n}} \quad (9)$$

Now, we consider a second ellipse  $E_2$  with the shape parameters:  $a_2 > 0$  is the semi-major axis,  $b_2 > 0$  is the semi-minor axis and  $C_2 = (0, d)$  is the ellipse center. So, let  $X_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$  be a point in  $\mathbb{R}^2$  and  $M_2$  a matrix given by

$$M_2 = \begin{pmatrix} \frac{1}{a_2^2} & 0 \\ 0 & \frac{1}{b_2^2} \end{pmatrix}$$

We write the equation of the ellipse  $E_2$  in a matrix form as follows

$$\phi_2(X_2) = (X_2 - C_2)^t M_2 (X_2 - C_2) = 1 \quad (10)$$

Now, we are looking for the expression of  $X_2$  that verifies Eq.10. For that propose, we consider  $n'$  as a normal vector to the tangent line  $T'$  of the ellipse  $E_2$  at  $X_2$ . Then, analogous to the previous case,

$$\frac{1}{2} \nabla \phi_2(X_2) = M_2 (X_2 - C_2) = \alpha_2 n', \quad (11)$$

with  $\alpha_2 > 0$ . If we consider that  $T = T'$ , then  $n = \pm n'$  and we say that  $T$  is a bi-tangent line to  $E_1$  and  $E_2$ . If in addition we consider that  $T$  is an uncrossed bi-tangent, then  $n = n'$  and

$$\frac{1}{2} \nabla \phi_2(X_2) = M_2 (X_2 - C_2) = \alpha_2 n \quad (12)$$

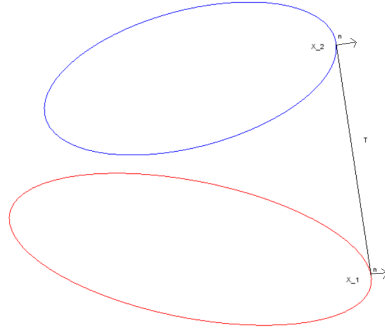


Figure 6: Normal vector  $n$  to the uncrossed bi-tangent  $T$ .

By using that  $M_2$  is an invertible matrix and from Eq.12, we obtain

$$X_2 = \alpha_2 M_2^{-1} n + C_2 \quad (13)$$

By substituting Eq.12 in Eq.10 and using that  $M_2$  is a diagonal matrix, one has

$$1 = (X_2 - C_2)^t M_2 (X_2 - C_2) = (\alpha_2 M_2^{-1} n)^t M_2 (\alpha_2 M_2^{-1} n) = \alpha_2^2 n^t M_2^{-1} n = \alpha_2^2 n^t M_2^{-1} n,$$

and from this we get

$$\alpha_2 = \frac{1}{\sqrt{n^t M_2^{-1} n}} \quad (14)$$

The normal vector  $n$  to the tangent line  $T$  is orthogonal to the vector  $X_2 - X_1$ , then we get that

$$n^t (X_2 - X_1) = 0 \quad (15)$$

and substituting Eq.8 and Eq.13 in this expression, one gets

$$n^t (\alpha_2 M_2^{-1} n - \alpha_1 M_1^{-1} n + C_2) = 0$$

Then, one has by Eq.9, Eq.14 and Eq.15

$$n^t \left( \frac{1}{\sqrt{n^t M_2^{-1} n}} M_2^{-1} n - \frac{1}{\sqrt{n^t M_1^{-1} n}} M_1^{-1} n + C_2 \right) = 0,$$



and simplifying, one gets

$$\sqrt{n^t M_2^{-1} n} - \sqrt{n^t M_1^{-1} n} + n C_2 = 0 \quad (16)$$

We consider that the normal vector to the uncrossed bi-tangent  $T$  at right is  $n = \begin{pmatrix} 1 \\ k \end{pmatrix}$ , where  $k \in \mathbb{R}$  (see Figure 6), and then, that the normal vector to the uncrossed bi-tangent at left is  $n = \begin{pmatrix} -1 \\ k \end{pmatrix}$ . By replacing both normal vectors in Eq. 16, we obtain the same result

$$\sqrt{\begin{pmatrix} 1 \\ k \end{pmatrix} \begin{pmatrix} a_2^2 \\ b_2^2 k \end{pmatrix}} - \sqrt{\begin{pmatrix} 1 \\ k \end{pmatrix} \begin{pmatrix} a_1^2 \\ b_1^2 k \end{pmatrix}} + \begin{pmatrix} 1 \\ k \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix} = 0$$

and then, one gets

$$\sqrt{a_2^2 + k^2 b_2^2} - \sqrt{a_1^2 + k^2 b_1^2} + d k = 0$$

We define the following function  $F_+$  depending in the variable  $k$

$$F_+(k) = \sqrt{a_2^2 + k^2 b_2^2} - \sqrt{a_1^2 + k^2 b_1^2} + d k, \quad (17)$$

and we conclude that Eq.15 is equivalent to the following equation

$$F_+(k) = 0 \quad (18)$$

By the method of Newton, we admit that this equation has at most two roots which will be determine the coordinates of the different points of tangents existing in the uncrossed case.

Now, we consider that  $T$  is a crossed bi-tangent of  $E_1$  and  $E_2$  at  $X_1$  and  $X_2$  respectively. In this case, the normal vector  $n'$  in Eq. 11 verifies  $n' = -n$ .

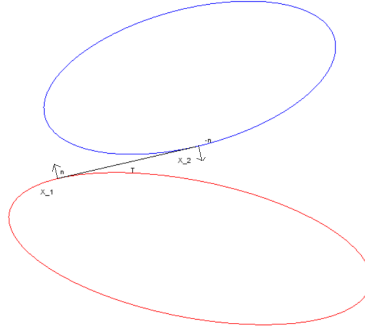


Figure 7: Normal vectors  $n$  and  $-n$  to the crossed bi-tangent  $T$ .

By following a similar process as the previous case, we obtain that

$$X_2 = -\alpha_2 M_2^{-1} n + C_2, \quad (19)$$

and this representation of  $X_2$  allows us to obtain the identity

$$n^t (-\alpha_2 M_2^{-1} n - \alpha_1 M_1^{-1} n + C_2) = 0$$

By replacing Eq.6, Eq. 9, Eq. 10 and Eq. 14 in the previous equation, and by considering  $n = \begin{pmatrix} k \\ 1 \end{pmatrix}$ , we obtain the equation

$$-\sqrt{\begin{pmatrix} k \\ 1 \end{pmatrix} \begin{pmatrix} a_2^2 k \\ b_2^2 \end{pmatrix}} - \sqrt{\begin{pmatrix} k \\ 1 \end{pmatrix} \begin{pmatrix} a_1^2 k \\ b_1^2 \end{pmatrix}} + \begin{pmatrix} k \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix} = 0$$

and then, one gets

$$-\sqrt{k^2 a_2^2 + b_2^2} - \sqrt{k^2 a_1^2 + b_1^2} + d = 0 \quad (20)$$

We define the following function  $F_-$  depending in the variable  $k$

$$F_-(k) = -\sqrt{k^2 a_2^2 + b_2^2} - \sqrt{k^2 a_1^2 + b_1^2} + d, \quad (21)$$

and we conclude that Eq.20 is equivalent to the following equation

$$F_-(k) = 0 \quad (22)$$

By using again the method of Newton, we admit that this equation has at most two roots which will be determine the coordinates of the different points of tangents existing in the crossed case.

To simplify the notation in the above construction, we choose  $C_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ,  $C_2 = \begin{pmatrix} 0 \\ d \end{pmatrix}$  and ellipses with rotation angles equal to zero. In Scilab code that we built to obtain the bi-tangents equations of two rotated ellipses and their tangents points, we consider no-degenerate generals ellipses.

## 5 Numerical results

### 5.1 Algorithm

In this section, we present an algorithm based on the second approach, that was introduced in the above section. This algorithm needs the inputs:

- The centers of the ellipses  $C_1$  and  $C_2$ .
- The lengths of the axes of the two ellipses  $a_1$ ,  $b_1$  and  $a_2$ ,  $b_2$ .
- The angles of rotation of the ellipses  $\theta_1$  and  $\theta_2$ .

Our Scilab code works in function of matrix representations of ellipses. First, it defines the matrices  $M_1$  and  $M_2$  of two no-degenerate real ellipses  $E_1$  and  $E_2$ , and the functions  $F_+$  and  $F_-$  defined in Eq.17 and Eq.21. Second, it determines  $k$  that solves  $F_+(k) = 0$  to calculate the points of the ellipses on the uncrosses bi-tangents. Finally, it solves  $F_-(k) = 1$  and  $F_-(k) = -1$  to compute the points of the ellipses on the uncrosses bi-tangents.

### 5.2 Examples

Analogous to the previous examples, for the following two examples: first we present the inputs of the code, second we present the graphic representations and last the coordinates of the pairs of points for every bi-tangent.

**Example 5.1.**

$$E_1 : a_1 = 1, b_1 = 1, \theta_1 = -\frac{\pi}{16}, C_1 = (2, 1)$$

$$E_2 : a_2 = 2.2, b_2 = 1, \theta_2 = \frac{\pi}{16}, C_2 = (2, 3.5)$$

$$\begin{array}{ll} p_1 : (2.9291086, 0.630193) & p_2 : (4.1248853, 3.6344804) \\ p_3 : (1.1089519, 0.5460911) & p_4 : (-0.1154272, 2.9496144) \\ p_5 : (2.3056766, 1.9521354) & p_6 : (1.9521354, 2.546692) \\ p_7 : (1.5065492, 1.8697737) & p_8 : (3.4280197, 2.9598862) \end{array}$$

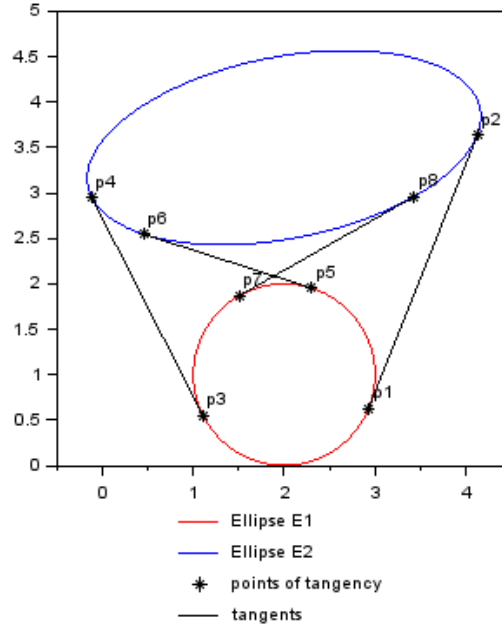


Figure 8: Example 5.1.

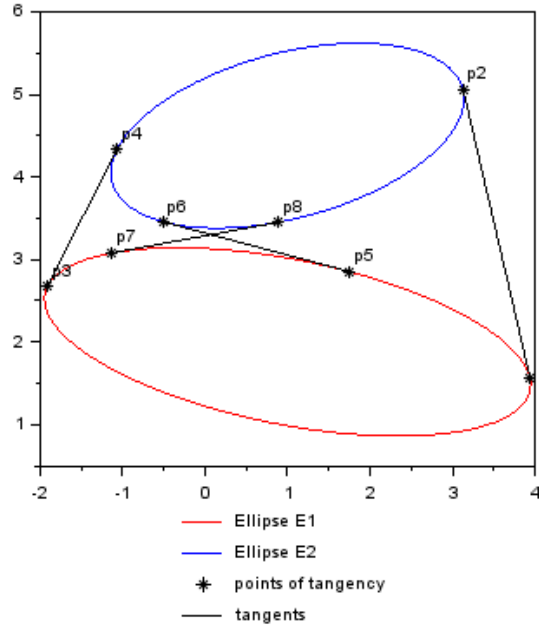


Figure 9: Example 5.2.

**Example 5.2.**

$$E_1 : a_1 = 3, b_1 = 1, \theta_1 = -\frac{\pi}{16}, C_1 = (1, 2)$$

$$E_2 : a_2 = 2.2, b_2 = 1, \theta_2 = \frac{\pi}{12}, C_2 = (1, 4.5)$$

$$\begin{array}{ll} p_1 : (3.938581, 1.5673923) & p_2 : (3.1287439, 5.0545741) \\ p_3 : (-1.9113075, 2.6742488) & p_4 : (-1.0650343, 4.3382995) \\ p_5 : (1.7515153, 2.8514694) & p_6 : (-0.5080026, 3.4546822) \\ p_7 : (-1.1297085, 3.0785714) & p_8 : (0.8874897, 3.4501697) \end{array}$$

## 6 Appendix

### 6.1 Appendix A: Algorithm Using Maple

The algorithm using maple and based on the first approach can be expressed as follows. We note that the input has to be entered in the 8th and the 9th commands of the algorithm.

```
>restart:
```

```
>with(plottools):with(plots):
```

```
>D1:=((2*m*p*sin(theta[1])^2+2*p*sin(theta[1])*cos(theta[1]))
*b[1]^2+(2*m*p*cos(theta[1])^2-2*p*sin(theta[1])*cos(theta[1]))
*a[1]^2)^2-(4*((cos(theta[1])^2+m^2*sin(theta[1])^2+2*m
*sin(theta[1])*cos(theta[1]))*b[1]^2+(sin(theta[1])^2+m^2
*cos(theta[1])^2-2*m*sin(theta[1])*cos(theta[1]))*(a[1]^2)))
*(b[1]^2*p^2*sin(theta[1])^2+a[1]^2*p^2*(cos(theta[1])^2)
-a[1]^2*b[1]^2):# calculates the first discriminant (eq(3))
in terms of m and p
```

```
>D2:=((2*m*(p-k)*sin(theta[2])^2+(2*(p-k))*sin(theta[2])
*cos(theta[2]))*b[2]^2+(2*m*(p-k)*cos(theta[2])^2-(2*(p-k))
*sin(theta[2])*cos(theta[2]))*a[2]^2)^2-(4*((cos(theta[2])^2
+m^2*sin(theta[2])^2+2*m*sin(theta[2])*cos(theta[2]))*b[2]^2
+(sin(theta[2])^2+m^2*cos(theta[2])^2-2*m*sin(theta[2])
*cos(theta[2]))*(a[2]^2)))*(b[2]^2*(p-k)^2*sin(theta[2])^2
+a[2]^2*(p-k)^2*(cos(theta[2])^2)-a[2]^2*b[2]^2):# calculates
the second discriminant (eq(4)) in terms of m and p
```

```
>x1:=(-(2*m*p*sin(theta[1])^2+2*p*sin(theta[1])*cos(theta[1]))
*b[1]^2+(2*m*p*cos(theta[1])^2-2*p*sin(theta[1])*cos(theta[1]))
*a[1]^2)/(2*((cos(theta[1])^2+m^2*sin(theta[1])^2
+2*m*sin(theta[1])*cos(theta[1]))*b[1]^2+(sin(theta[1])^2
+m^2*cos(theta[1])^2-2*m*sin(theta[1])*cos(theta[1]))*(a[1]^2)))):
# calculates the abscissas of the points of tangency of the
uncrossed tangent lines in terms of m and p
```

```
>x2:=(-(2*m*(p-k)*sin(theta[2])^2+(2*(p-k))*sin(theta[2])
*cos(theta[2]))*b[2]^2+(2*m*(p-k)*cos(theta[2])^2-(2*(p-k))
*sin(theta[2])*cos(theta[2]))*a[2]^2)/(2*((cos(theta[2])^2+m^2
*sin(theta[2])^2+2*m*sin(theta[2])*cos(theta[2]))*b[2]^2
+(sin(theta[2])^2+m^2*cos(theta[2])^2-2*m*sin(theta[2])
*cos(theta[2]))*(a[2]^2)))):# calculates the abscissas
of the points of tangency of the crossed tangent lines in
```

terms of  $m$  and  $p$

```
>y1:=m*x1+p;# calculates the ordinates of the points of  
tangency of the uncrossed tangent lines in terms of  $m$  and  $p$ 
```

```
>y2:=m*x2+p;# calculates the ordinates of the points of  
tangency of the crossed tangent lines in terms of  $m$  and  $p$ 
```

```
>theta[1]:=(1/3)*Pi; a[1]:=2; b[1]:=1;# chooses the  
parameters of the first ellipse
```

```
>theta[2]:=0; a[2]:=1; b[2]:=1; k:= 3;# chooses the  
parameters of the second ellipse
```

```
>simplify(D1): simplify(D2):
```

```
>S:=solve({D1=0, D2=0},[m, p]);# solves a system of two  
equations, the zero sets of the two discriminants, in two  
unknowns  $m$  and  $p$ 
```

```
>M:=map(allvalues, S);# presents the solutions
```

```
>E1:=ellipse([0, 0], a[1], b[1]);# defines the first ellipse
```

```
>E2:=ellipse([0, k], a[2], b[2]);# defines the second ellipse
```

```
>x11:=subs(M[1],x1); x21:=subs(M[1],x2); y11:=subs(M[1],y1);  
y21:=subs(M[1],y2);# substitutes the obtained values of  
 $m$  and  $p$  in the expressions of the abscissas and ordinates  
of the points of tangency of the uncrossed tangents to the  
first ellipse
```

```
>x12:=subs(M[2],x1); x22:=subs(M[2],x2); y12:=subs(M[2],y1);  
y22:=subs(M[2],y2);# substitutes the obtained values of  
 $m$  and  $p$  in the expressions of the abscissas and ordinates  
of the points of tangency of the uncrossed tangents to the  
second ellipse
```

```
>x13:=subs(M[3],x1); x23:=subs(M[3],x2); y13:=subs(M[3],y1);  
y23:=subs(M[3],y2);# substitutes the obtained values of  
 $m$  and  $p$  in the expressions of the abscissas and ordinates  
of the points of tangency of the crossed tangents to the  
first ellipse
```

```
>x14:=subs(M[4],x1); x24:=subs(M[4],x2); y14:=subs(M[4],y1);  
y24:=subs(M[4],y2);# substitutes the obtained values of  
 $m$  and  $p$  in the expressions of the abscissas and ordinates  
of the points of tangency of the crossed tangents to the  
second ellipse
```

```
>display(line([x11, y11], [x21, y21]),
```

```

line([x12, y12], [x22, y22]), line([x13, y13], [x23, y23]),
line([x14, y14], [x24, y24]), rotate(E1, theta[1]),
rotate(E2, theta[2], [0, k]), scaling = constrained);# draws
the two ellipses and the corresponding two bi-tangents

```

## 6.2 Appendix B: Algorithm Using Scilab

The algorithm using Scilab and based on the second approach can be expressed as follows.

```

a=3; b=1; th=%pi/16; c=[1;2]; con1="r"// Parameters Ellipse E1
a1=2.2; b1=1; th1=%pi/12; c1=[1;4.5];con2="b"//Parameters Ellipse E2

//function definition that draws a rotated ellipse
function DrawElip(a,b,th,c,co)
t=[0:0.1:6.38];
X=a*cos(t); Y=b*sin(t);
plot(c(1)+X*cos(th)-Y*sin(th),c(2)+Y*cos(th)+X*sin(th),co)
endfunction

function DrawLine(a,b) //function definition that draws a line
t=[0:0.02:1];
X=a*(1-t)+b*t;
plot(X(1,:),X(2,:), "black")
endfunction

//function definition that computes the rotation matrix of ellipses
function M=MatElip(a,b,th)
M=[1/a^2,0; 0,1/b^2];
Rot=[cos(th),-sin(th); sin(th), cos(th)];
M=Rot*M*Rot';
endfunction

axes=get("current_axes");//get the handle of the newly created axes
axes.isoview="on"; // isoview mode

M=MatElip(a,b,th); // computation of rotation matrix of E1
DrawElip(a,b,th,c,con1); // drawing of the ellipse E1
M1=MatElip(a1,b1,th1); // computation of rotation matrix of E2
DrawElip(a1,b1,th1,c1,con2); // drawing of the ellipse E2

// function definition of the inner product between a uncrossed
// bi-tangent and the normal vector to the tangent line of E1 and E2
function y=Fplus(k)
n=[n1;k]
y=(c1(2)-c(2))*k+sqrt(n'*(M1\ n))-sqrt(n'*(M\ n))
endfunction

// function definition of the inner product between a crossed
// bi-tangent and the normal vector to the tangent line of E1 and E2
function y=Fminus(k)
n=[k;1]

```

```

y=c1(2)-c(2)-sqrt(n'*(M1\ n))-sqrt(n'*(M\ n))
endfunction
// Compute of tangents points to the uncrossed bi-tangent at right
n1=1
k=fsolve(0,Fplus)
n=[1;k]
alp=sqrt(n'*(M\ n))
x=c+(M\ n)/alp
plot(x(1),x(2),'black*')
alp=sqrt(n'*(M1\ n))
x1=c1+(M1\ n)/alp
plot(x1(1),x1(2),'black*')
DrawLine(x,x1)
xstring(x(1),x(2),"p1")
xstring(x1(1),x1(2),"p2")

// Compute of tangents points to the uncrossed bi-tangent at left
n1=-1
k=fsolve(0,Fplus)
n=[-1;k]
alp=sqrt(n'*(M\ n))
x=c+(M\ n)/alp
plot(x(1),x(2),'black*')
alp=sqrt(n'*(M1\ n))
x1=c1+(M1\ n)/alp
plot(x1(1),x1(2),'black*')
DrawLine(x,x1)
xstring(x(1),x(2),"p3")
xstring(x1(1),x1(2),"p4")

// Compute of tangents points to the crossed bi-tangent to the
// right of E1 and to the left of E2
k=fsolve(1,Fminus)
n=[k;1]
alp=sqrt(n'*(M\ n))
x=c+(M\ n)/alp
plot(x(1),x(2),'black*')
n=-[k;1]
alp=sqrt(n'*(M1\ n))
x1=c1+(M1\ n)/alp
plot(x1(1),x1(2),'black*')
DrawLine(x,x1)
xstring(x(1),x(2),"p5")
xstring(x1(1),x1(2),"p6")

// Compute of tangents points to the crossed bi-tangent to the
// right of E2 and to the left of E1
k=fsolve(-1,Fminus)
n=[k;1]
alp=sqrt(n'*(M\ n))
x=c+(M\ n)/alp

```

```

plot(x(1),x(2),'black*')
n=-[k;1]
alp=sqrt(n*(M1\ n))
x1=c1+(M1\ n)/ alp
plot(x1(1),x1(2),'black*')
DrawLine(x,x1)
xstring(x(1),x(2),"p7")
xstring(x1(1),x1(2),"p8")

legends(['Ellipse E1' 'Ellipse E2'
'points of tangency' 'tangents'],[5,2,-10,1],with_box=0f,opt=6)

```

## References

- [1] I. Emiris, E. Tsigaridas and G. Tzoumas, *Predicates for the exact Voronoi diagram of ellipses under the Euclidean metric*, Int. J. Comput. Geom. Appl. (2008) 18(6) 567-597.
- [2] L. Habert and M. Pocchiola, *Computing the convex hull of disks using only their chirotope*, European Workshop on Computational Geometry (2004).
- [3] L. Habert, *Computing bitangents for ellipses*, In Proc. 17th Canad. Conf. Comp. Geom. (2005) 294-297.